Summary of helicity supertraces (relevant for us)

- Supersymmetric theories in asymptotic $\mathbb{R}^{1,3}$
- Extended susy $\mathcal{N}=2,4,8$.
- States (reps) labelled by Mass M, central charge Z w/ $M \geq |Z|$.
- (Class of special states) BPS states preserve fraction of susy

Focus on BPS states preserving 4 susy

Theory	# of susy	Central charge	Fraction of preserved susy	BPS rule	Relevant helicity supertrace	Type II / M6
$\mathcal{N}=8$	32	Z_1, Z_2, Z_3, Z_4	1/8 BPS	$M=\|Z_1\|>\|Z_i\|$.	B_{14}	T^6
$\mathcal{N}=4$	16	Z_1, Z_2	1/4 BPS	$M=\|Z_1\|>\|Z_2\|$	B_6	$K3\times T^2$
$\mathcal{N}=2$	8	Z	1/2 BPS	$M=\|Z\|$	B_2	CY_3

Susy index → helicity supertrace

$$B_{2n} = \mathrm{Tr}\,(-1)^{2J_3}\,(2J_3)^{2n}$$

✓ receives contribution from states breaking $4n$ susy

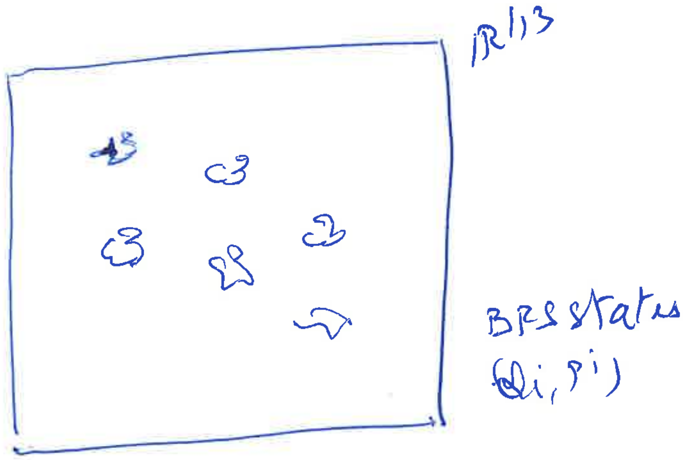
NO " " " " $>4n$ "

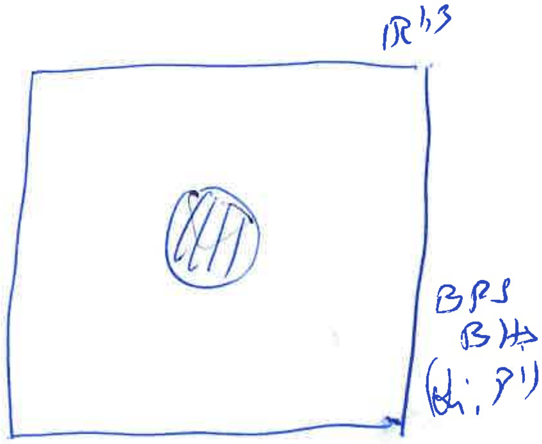
→ In this sense index "counts" BPS states.

~~T^6 $K3\times T^2$ CY_3 ↑ Easier to count the index in string theory ↓ Simpler (Smaller) description of (smaller effective field theory).~~

Easier to count ↑ T^6 $K3\times T^2$ CY_3 ↓ smaller effective field theory

§ BPS States & BPS BHs





Microscopic ($g_s \ll 1$)

- String theory IIA/M_6.
- Enumerate strings, branes wrapping cycles, ...

  $\Rightarrow$ all BPS states w/ charges (Q_i, P^i)
- point-like in 4d

- e.g. $\frac{1}{2}$-BPS states in $\mathcal{N}=2$ theory $M = |Z|$.
- Calculate B_2.

Macroscopic $g_s \gg 1$

- Effective theory $(g_{\mu\nu}, A_\mu^i, \varphi^a, \psi^\alpha)$
- Find supersymmetric BH solutions.

- e.g. $\frac{1}{2}$-BPS BHs in $\mathcal{N}=2$ supergravity w/ $M_{(\infty)} = |Z_\infty|$.
- Measure entropy.

Note: Q_i, P^i specify states/solns. completely

$\Rightarrow Z = Z(Q_i, P^i)$

Simplest BH soln (BPS): 1 electric charge Q, $Z = Q$.

BPS $\Rightarrow M = Q$

$\rightarrow$ extremal BHs

§ Charged & extremal BHs

Einstein-Maxwell theory

$$S_{E-M} = \int d^4x \sqrt{g} \left(\frac{R}{16\pi G} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right)$$

Reissner-Nördstrom BH soln.

$$ds^2 = -f(\rho)\,dt^2 + \frac{d\rho^2}{f(\rho)} + \rho^2 d\Omega_2^2$$

$G=1$ $\qquad F_{\rho t} = Q/\rho^2 \quad , \quad f(\rho) = \left(1-\frac{\rho_+}{\rho}\right)\left(1-\frac{\rho_-}{\rho}\right)$

$$T_{BH} = \frac{1}{4\pi} f'(\rho_+) \qquad \left.\begin{array}{l} \rho_+ + \rho_- = 2M \\ \rho_+ \rho_- = Q^2 \end{array}\right\} \quad \rho_\pm = M \pm \sqrt{M^2 - Q^2}$$

Extremal BH: $\rho_+ = \rho_- \Rightarrow M = Q, \quad \rho_\pm = Q$

$$ds^2 = -\left(1-\frac{Q}{\rho}\right)^2 dt^2 + \frac{d\rho^2}{(1-Q/\rho)^2} + \rho^2 d\Omega^2$$

$$\xrightarrow[\rho = Q + r]{} \; = -\frac{dt^2}{(1+Q/r)^2} + (1+Q/r)^2 \left(dr^2 + r^2 d\Omega_2^2\right)$$

$\boxed{Ex}$

Horizon at $r=0$.

As $r \to 0$, $\qquad ds^2 = Q^2 \Big(\underbrace{-r^2 d\tilde{t}^2 + \frac{dr^2}{r^2}}_{AdS_2} + \underbrace{d\Omega_2^2}_{S^2} \Big)$

$\tilde{t} = t/Q$

Note $\quad T_{BH} \xrightarrow[\rho_+ \to \rho_-]{} 0 \quad , \quad S_{BH} = \pi Q^2$

Extremal RN BH:

- $T=0$
- $M=Q$
- Near-horizon $AdS_2 \times S^2$
- $S_{BH} = \pi Q^2 \longrightarrow \pi(Q^2+P^2)$

Introduce magnetic charge

Near-horizon configuration $AdS_2 \times S^2$ is a classical soln. to Einstein-Maxwell theory in its own right.

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§ $\mathcal{N}=2$ pure supergravity $(g_{\mu\nu}, A_\mu, \psi^i_\mu)$ ← gravitini.

$$S = S_{E-M} + \frac{1}{16\pi}\int d^4x\,\left(\bar{\psi}_\mu \cancel{D}^{\mu\nu}\psi_\nu + \bar{\psi}_\mu F^{\mu\nu}\psi_\nu + \psi_\mu F^{\mu\nu}\psi_\nu + \text{higher order}\right).$$

Gravity : $M, (P_\mu)$ measured asymptotically through Killing vector ∂_μ

Supergravity supercharges measured asymptotically through Killing spinor ε. → $\delta_\varepsilon \psi_\mu = 0$.

Condition of susy

$g_{\mu\nu} = e^a_\mu e^a_\nu$.

$$\delta_\varepsilon e^a_\mu = \bar{\varepsilon}\gamma^a\psi_\mu$$

$$\delta_\varepsilon \psi_\mu = D_\mu\varepsilon + \ldots$$

$$\quad \text{“} \; (\partial_\mu + \tfrac{1}{2}\omega^{ab}_\mu\gamma^{ab})\varepsilon + \ldots$$

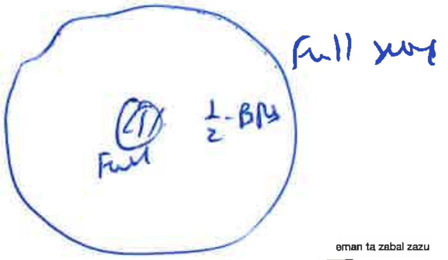
$$\{\delta_\varepsilon, \delta_{\bar{\varepsilon}}\} = \bar{\varepsilon}^{\dot\alpha}\gamma^\mu_{\alpha\dot\alpha}\varepsilon^{\alpha}\partial_\mu$$

$\mathcal{N}=1$ susy algebra

$\mathcal{N}=2$ susy ε^i $i=1,2$ → ~~[illegible]~~ → $\mathcal{N}=2$ susy algebra
Transf. δ_{ε^i} realize → w/ $Z = Q$.

Special solns of bosonic theory admit Killing spinors.
↳ supersymmetric solns.

e.g. $\mathbb{R}^{1,3}$: admits 8 Killing spinors] Full-BPS
$AdS_2\times S^2$: " " " "
Extreme RN soln : " 4 Killing ".] $\frac{1}{2}$-BPS



§ Embedding in string theory

[cf Freedman + van Proeyen "Supergravity" Chap 22.]

Type II / CY_3 $\longrightarrow$ 4d $\mathcal{N}=2$ sugra + n_V vector multiplets + hyper multiplets.

Field multiplets of $\mathcal{N}=2$ susy:

graviton multiplet : $(g_{\mu\nu}, A_\mu^0, \psi_\mu)$

vector " $(A_\mu^a, \phi^a, \bar\phi^a, \lambda^a)$ $a = 1, 2, \ldots n_V$.

[Hypermultiplets do not participate in BH solns.]

$$S = \frac{1}{16\pi}\int d^4x \sqrt{-g}\Big(R - f^{IJ}(\phi^a) F_{\mu\nu}^I F^{\mu\nu J} - \tilde f^{IJ}(\phi) F_{\mu\nu}^I \tilde F^{\mu\nu J} - G_{a\bar b}(\phi)\partial_\mu \phi^a \partial^\mu \bar\phi^b + \text{fermions}\Big).$$

- Kinetic mixing $I = 0, 1, \ldots n_V$.
- Scalar fields involved in BH soln. $\Rightarrow$ complicated [cf R-N $(g_{\mu\nu}, A_\mu)$]
- n_V, f^{IJ}, $\tilde f^{IJ}$, $G_{a\bar b}$ all determined by data of CY_3.
- ~~Central char~~ ~~Action is $\mathcal{N}=2$ susy~~ w/ ~~$Z = Z(q_I^0, p^I), \phi$~~

Action is $\mathcal{N}=2$ susy w/ central charge $Z(q_I, p^I, \phi^a, \bar\phi^a)$.

Puzzle: $\phi^a, \bar\phi^a$ free at ∞ $\rightarrow$ "moduli", no potential.

$\Rightarrow$ can change continously

v/s $d_{BH} \in \mathbb{Z}$ cannot change continously.

§ Attractor mechanism. Consider BH w/ charges (q_I, p^I).

~~ds^2~~ = Spherical symmetry.

$$ds^2 = -e^{2u(r)}dt^2 + e^{-2u(r)}(dr^2 + r^2 d\Omega_2^2)$$

$r = 1/s$ $\quad s = \infty \rightarrow$ horizon $\quad s = 0 \rightarrow$ asymptotic ∞.

Eqns of susy $\overset{\text{[Jump, cf Textbook Eqs]}}{\Longrightarrow}$ effective 1d theory governed by :

$$S_{eff} = \int_{\infty}^{0} ds \left(\dot{U}^2 + G_{a\bar{b}} \dot{\phi}^a \dot{\bar{\phi}}^b + e^{2U} \underbrace{V_{BH}}(\phi, \bar{\phi}) \right).$$

↳ function of $f, \bar{f}$, etc

+ Energy is conserved ($E = 0$.).

⇒ 1^{st} order eqns. for U & ϕ^a.

- Scalars develop a potential near BH horizon
- Flow eqn (1^{st} order) $s \to \infty$ = fixed pt.

⇒ Near horizon, $\phi^a \xrightarrow[r\to 0]{} \phi^a_*(Q, P)$, $e^{-2U} \underset{r\to 0}{\approx} \frac{\overbrace{Z_*(Q, P; \phi^{\bar{a}}_*, \bar{\phi}^{\bar{a}}_*)}^{Z_*(Q,P)}}{r}$

[soln for fixed pt.] →

$$\Rightarrow ds^2 \underset{r\to 0.}{\sim} |Z_*(Q,P)|^2 \left(\underbrace{-r^2 dt^2 + \frac{dr^2}{r^2}}_{AdS_2} + \underbrace{d\Omega_2^2}_{S^2} \right).$$

$$S_{BH} = \pi |Z_*(Q, P)|^2$$

- Effectively, we have done a rotation in field space s.t. Problem is governed by 1 gauge field w/ charge Z_*. & no scalars

⇒ Basic R-N soln.
Just like